INDEPENDENCE OF SETS WITHOUT STABILITY

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Abstract. We presents an independence relation on sets, one can define dimension by it, assuming that we have an abstract elementary class with a forking notion that satisfies the axioms of a good frame minus stability.

Contents

Introduction. We would like to find an "independence relation" such that if K is a class of models, $M_0, M_1 \in K$ and $J \subseteq M_1 - M_0$ then we can say if J is independent in M_1 over M_0 or not. The independence relation should satisfy the following things:

(1) If K is the class of fields with character 0, then independence is linear independence.
(2) If $J^* \subseteq J \subseteq M_1 - M_0$ and J is independent in M_1 over M_0 then J^* is independent in M_1 over M_0.
(3) If J_1, J_2 are maximal independent subsets of M_1 over M_0 then $|J_1| = |J_2|$ or they both finite.

It is hard to find such a relation, so we decrease our ambitions in two aspects:

(1) K should be a "good class" of models.
(2) M_0 should be a "good submodel" of M_1.

The restriction of the context is made in two steps: In section 1, we define a list of axioms, a pair $(K, \preceq_{\mathfrak{k}})$ that satisfy those axioms is called an "abstract elementary class" (in short a.e.c.). In section 2 we define a list of axioms of a non-forking relation and restrict the study to those a.e.c.'s, one can find a non-forking relation related to them. On such a.e.c.'s we exhibit in section

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3 an independence relation and prove that one can define dimension by it. Sections 1,2,3 are self contained. In section 4 assuming existence of uniqueness triples (so familiarity with [JrSh 875], [Sh 600] or [Sh 705] is assumed), We prove that the relations independence and finitely independence are the same and more properties.

What are the connections between the present paper and other papers? In [JrSh 875] we study stability theory without assuming stability, but weak stability. The main purpose is to study abstract elementary classes (shortly a.e.c.'s) which are $PC_{\aleph_0}$. The theorems we prove here, may be useful in the study of such classes too.

The frame we define ("good frame minus stability") is similar to the weak forking notion which is defined in [GrKo], which is parallel to simple first order theories, see Remark 1.5 on page 8 of [GrKo] (but there is a significant difference: here we work in one cardinality only).

We define independence as in [Sh 705]. In what aspects do we improve here the results in section 5 of [Sh 705]?

(1) We do not assume stability.
(2) We do not assume successfulness.
(3) We do not assume $goodness^+$.
(4) We prove several important propositions without assuming that $K^{3,uq}$ has existence.

1. Abstract Elementary Classes

Definition 1.1 (Abstract Elementary Classes).

(1) Let K be a class of models for a fixed vocabulary and let $\preceq=\preceq_{\mathfrak{k}}$ be a 2-place relation on K. The pair $\mathfrak{k}=(K,\preceq_{\mathfrak{k}})$ is an *a.e.c.* if the following axioms are satisfied:
   (a) $K,\preceq$ are closed under isomorphisms. In other words, if $M_1\in K$, $M_0\preceq_{\mathfrak{k}} M_1$ and $f:M_1\to N_1$ is an isomorphism then $N_1\in K$ and $f[M_0]\preceq_{\mathfrak{k}} N_1$.
   (b) $\preceq$ is a partial order and it is included in the inclusion relation.
   (c) If $\langle M_\alpha:\alpha<\delta\rangle$ is a continuous $\preceq_{\mathfrak{k}}$-increasing sequence, then
   $$M_0\preceq\bigcup\{M_\alpha:\alpha<\delta\}\in K.$$
   (d) Smoothness: If $\langle M_\alpha:\alpha<\delta\rangle$ is a continuous $\preceq_{\mathfrak{k}}$-increasing sequence, and for every $\alpha<\delta$, $M_\alpha\preceq N$, then
   $$\bigcup\{M_\alpha:\alpha<\delta\}\preceq N.$$
   (e) If $M_0\subseteq M_1\subseteq M_2$ and $M_0\preceq M_2\wedge M_1\preceq M_2$, then $M_0\preceq M_1$.
   (f) There is a Lowenheim Skolem Tarski number, $LST(\mathfrak{k})$, which is the minimal cardinal λ, such that for every model $N\in K$ and a subset A of it, there is a model $M\in K$ such that $A\subseteq M\preceq N$ and the cardinality of M is $\leq\lambda+|A|$.

(2) $\mathfrak{k} = (K, \preceq)$ is an *a.e.c. in* λ if: The cardinality of every model in K is λ, and it satisfies axioms a,b,d,e of a.e.c., and for sequences $\langle M_\alpha : \alpha < \delta \rangle$ with $\delta < \lambda^+$ it satisfies axiom c too.

In [Gr 21] there are examples of a.e.cs. The following are examples of naturals classes which are not a.e.cs.

Example 1.2. The class of *sets* (i.e. models without relations or functions) with cardinality less than κ, where $\aleph_0 \leq \kappa$ and the relation is $\subseteq$, is *not* an a.e.c., as it does not satisfy axiom c.

The class of sets with the relation $\preceq = \{(M, N) : M \subseteq N \text{ and } ||N - M|| > \kappa\}$ where $\aleph_0 \leq \kappa$, is not an a.e.c., as it does *not* satisfy smoothness (axiom d).

Definition 1.3.

(1) For a model $M \in K$ we denote its universe by $|M|$, and its cardinality by $||M||$.
(2) $K_\lambda =: \{M \in K : ||M|| = \lambda\}$.

Definition 1.4.

(1) Let M, N be models in K and let f be an injection of M to N. We say that f is a $\preceq_{\mathfrak{k}}$*-embedding*, or f is an embedding (if $\preceq_{\mathfrak{k}}$ is clear from the context), when f is an injection with domain M and $Im(f) \preceq_{\mathfrak{k}} N$.
(2) A function $f : B \to C$ is *over* A, if $A \subseteq B \bigcap C$ and $x \in A \Rightarrow f(x) = x$.

Definition 1.5.

(1) $K^3 =: \{(M, N, a) : M \in K,\ N \in K,\ M \preceq N,\ a \in N\}$.
(2) $K^3_\lambda =: \{(M, N, a) : M \in K_\lambda,\ N \in K_\lambda,\ M \preceq N,\ a \in N\}$.
(3) $E^* = E^*_k$ is the following relation on K^3: $(M_0, N_0, a_0)E^*(M_1, N_1, a_1)$ iff $M_1 = M_0$ and for some N_2, f we have: $N_1 \preceq N_2,\ f : N_0 \to N_2$ is an embedding over M_0 and $f(a_0) = a_1$.
(4) $E^*_\lambda := E^* \restriction K^3_\lambda$.
(5) $E = E_k$ is the closure of E^* under transitivity, i.e. the closure to an equivalence relation.

Definition 1.6.

(1) We say that $\mathfrak{k}_\lambda$ has *amalgamation* when: For every M_0, M_1, M_2 in K_λ, such that $n < 3 \Rightarrow M_0 \preceq_{\mathfrak{k}} M_n$, for some (f_1, f_2, M_3) we have: $f_n : M_n \to M_3$ is an embedding over M_0, i.e. the diagram below commutes. In such a case we say that that (f_1, f_2, M_3) is an amalgamation of M_1, M_2 over M_0 or that M_3 is an amalgam of M_1, M_2 over M_0.

$$\begin{tikzcd} M_1 \arrow[r, "f_1"] & M_3 \\ M_0 \arrow[u, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & M_2 \arrow[u, "f_2"'] \end{tikzcd}$$

(2) we say that K_λ has *joint embedding* when: If $M_1, M_2 \in K_\lambda$, then there are f_1, f_2, M_3 such that for $n = 1, 2$ $f_n : M_n \to M_3$ is an embedding and $M_3 \in K_\lambda$.
(3) $M \in K$ is $\preceq_{\mathfrak{k}}$-*maximal* if there is no $N \in K$ such that $M \prec N$.

Proposition 1.7.

(1) $(M, N_0, a)E^*(M, N_1, b)$ iff there is an amalgamation N, f_0, f_1 of N_0 and N_1 over M such that $f_0(a) = f_1(b)$.
(2) E^* is a reflexive, symmetric relation.
(3) If $\mathfrak{k}$ has amalgamation, then E^* is an equivalence relation.
(4) If $\mathfrak{k}_\lambda$ has amalgamation, then E^*_λ is an equivalence relation.

Proof. Easy. $\dashv$

Definition 1.8.

(1) For $(M, N, a) \in K^3$ let $tp(a, M, N) = tp_k(a, M, N)$, the *type* of a in N over M, be the equivalence class of (M, N, a) under E (In other texts, it is called "$ga - tp(a/M, N)$").
(2) $S(M) = S_k(M) = \{tp(a, M, N) : (M, N, a) \in K^3\}$.
(3) If $M_0 \preceq M_1, p \in S(M_1)$ then define $p \restriction M_0 = tp(a, M_0, N)$, (by the definitions of E, E^* it is easy to check that $p \restriction M_0$ does not depend on the representative of p).
(4) If $p = tp(a, M, N)$ and $f : M \to M^*$ is an isomorphism, then we define $f(p) := tp(f(a), f[M], f^+[N])$ where f^+ is an extension of f ($f(p)$ does not depend on the choice of f^+).

2. Good Frame Minus Stability

We define the non-forking frame, we are going to work with. It is similar to the good frame that Shelah defined in section 2 of [Sh 600], but here we do not assume basic stability.

Definition 2.1. $\mathfrak{s} = (\mathfrak{k}, S^{bs}, \bigcup)$ is a *good λ-frame minus stability* if:
(1) $\mathfrak{k} = (K, \preceq_{\mathfrak{k}})$ is an a.e.c., $LST(\mathfrak{k}) \leq \lambda$, and the following four axioms are satisfied in $\mathfrak{k}_\lambda$: It has joint embedding, amalgamation and there is no $\preceq$-maximal model in $\mathfrak{k}_\lambda$.
(2) S^{bs} is a function with domain K_λ, which satisfies the following axioms:
(a) It respects isomorphisms.
(b) $S^{bs}(M) \subseteq S^{na}(M) =: \{tp(a, M, N) : M \prec N \in K_\lambda,\ a \in N - M\}$.
(c) Densite of basic types: If $M \prec N$ in K_λ, then there is $a \in N - M$ such that $tp(a, M, N) \in S^{bs}(M)$.

(3) the relation $\bigcup$ satisfies the following axioms:

(a) $\bigcup$ is a subset of $\{(M_0, M_1, a, M_3) : n \in \{0,1,3\} \Rightarrow M_n \in K_\lambda,\ a \in M_3 - M_1,\ n < 2 \Rightarrow tp(a, M_n, M_3) \in S^{bs}(M_n)\}$.

(b) Monotonicity: If $M_0 \preceq M_0^* \preceq M_1^* \preceq M_1 \preceq M_3,\ M_1^* \bigcup \{a\} \subseteq M_3^{**} \preceq M_3^*$, then $\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M_0^*, M_1^*, a, M_3^{**})$. [So we can say "$p$ does not fork over M_0" instead of $\bigcup(M_0, M_1, a, M_3)$].

(c) Local character: If $\langle M_\alpha : \alpha \leq \delta \rangle$ is an increasing continuous sequence, and $tp(a, M_\delta, M_{\delta+1}) \in S^{bs}(M_\delta)$, then there is $\alpha < \delta$ such that $tp(a, M_\delta, M_{\delta+1})$ does not fork over M_α.

(d) Uniqueness of the non-forking extension: If $p, q \in S^{bs}(N)$ do not fork over M, and $p \restriction M = q \restriction M$, then p=q.

(e) Symmetry: If $M_0 \preceq M_1 \preceq M_3,\ a_1 \in M_1,\ tp(a_1, M_0, M_3) \in S^{bs}(M_0)$, and $tp(a_2, M_1, M_3)$ does not fork over M_0, *then* for some $M_2, M_3^*,\ a_2 \in M_2,\ M_0 \preceq M_2 \preceq M_3^*,\ M_3 \preceq M_3^*$, and $tp(a_1, M_2, M_3^*)$ does not fork over M_0.

(f) Existence of non-forking extension: If $p \in S^{bs}(M)$ and $M \prec N$, then there is a type $q \in S^{bs}(N)$ such that q does not fork over M and $q \restriction M = p$.

(g) Continuity: Let $\langle M_\alpha : \alpha \leq \delta \rangle$ be an increasing continuous sequence. Let $p \in S(M_\delta)$. If for every $\alpha \in \delta,\ p \restriction M_\alpha$ does not fork over M_0, then $p \in S^{bs}(M_\delta)$ and does not fork over M_0.

Proposition 2.2 (versions of axiom f)**.** *If for $n < 3$ $M_n \in K_\lambda$, $M_0 \preceq M_n$, and $tp(a, M_1, M_0) \in S^{bs}(M_0)$ then:*

(1) There are M_3, f such that:
  (a) $M_2 \preceq M_3$.
  (b) $f : M_1 \to M_3$ is an embedding over M_0.
  (c) $tp(f(a), M_2, M_3)$ does not fork over M_0.

(2) There are M_3, f such that:
  (a) $M_1 \preceq M_3$.
  (b) $f : M_2 \to M_3$ is an embedding over M_0.
  (c) $tp(a, f[M_2], M_3)$ does not fork over M_0.

Proof. Easy. ⊣

Proposition 2.3 (transitivity)**.** *Suppose $\mathfrak{s}$ is a semi-good λ-frame minus stability. If $M_0 \preceq M_1 \preceq M_2,\ p \in S^{bs}(M_2)$ does not fork over $M_1,\ p \restriction M_1$ does not fork over M_0, then p does not fork over M_0.*

Proof. Suppose $M_0 \prec M_1 \prec M_2,\ n < 3 \Rightarrow M_n \in K_\lambda,\ p_2 \in S^{bs}(M_2)$ does not fork over M_1 and $p_2 \restriction M_1$ does not fork over M_0. For $n < 2$ define $p_n = p_2 \restriction M_n$. By Definition 2.1.f there is a type $q_2 \in S^{bs}(M_2)$ such that $q_2 \restriction M_0 = p_0$ and q_2 does not fork over M_0. Define $q_1 = q_2 \restriction M_1$. By Definition 2.1.b (monotonicity) q_1 does not fork over M_0. So by Definition 2.1.d (uniqueness) $q_1 = p_1$. Using again Definition 2.1.f, we get $q_2 = p_2$, as they do not fork over M_1. By the definition of q_2 it does not fork over M_0. ⊣

Fact 2.4. *Suppose*

1. *$\mathfrak{s}$ is a semi-good λ-frame minus stability.*
2. *$n < 3 \Rightarrow M_0 \preceq M_n$.*
3. *For $n = 1, 2$, $a_n \in M_n - M_0$ and $tp(a_n, M_0, M_n) \in S^{bs}(M_0)$.*

Then there is an amalgamation M_3, f_1, f_2 of M_1, M_2 over M_0 such that for $n = 1, 2$ $tp(f_n(a_n), f_{3-n}[M_{3-n}], M_3)$ does not fork over M_0.

For completeness we give a proof:

Proof. Suppose for $n = 1, 2$ $M_0 \prec M_n \wedge tp(a_n, M_0, M_n) \in S^{bs}(M_0)$. By Proposition 2.2.1, there are N_1, f_1 such that:

1. $M_1 \preceq N_1$.
2. $f_1 : M_2 \to N_1$ is an embedding over M_0.
3. $tp(f_1(a_2), M_1, N_1)$ does not fork over M_0.

By Definition 2.1.f (the symmetry axiom), there are a model N_2, $N_1 \preceq N_2 \in K_\lambda$ and a model $N_2^* \in K_\lambda$ such that: $M_0 \bigcup \{f_1(a_2)\} \subseteq N_2^* \preceq N_2$ and $tp(a_1, N_2^*, N_2)$ does not fork over M_0.
By Claim 2.2.2 (substituting N_2^*, N_2, N_2, a_1 which appear here instead of M_0, M_1, M_2, a there) there are N_3, f_2 such that:

1. $N_2 \preceq N_3$.
2. $f_2 : N_2 \to N_3$ is an embedding over N_2^*.
3. $tp(a_1, f_2[N_2], N_3)$ does not fork over N_2^*.

So by Claim 2.3 (page 5), $tp(a_1, f_2[N_2], N_3)$ does not fork over M_0. So as $M_0 \preceq f_2 \circ f_1[M_2] \preceq f_2[N_2]$ by Definition 2.1.b (monotonicity) $tp(a_1, f_2 \circ f_1[M_2], N_3)$ does not fork over M_0. As $f_2 \restriction N_2^* = id_{N_2^*}$, $f_2(f_1(a_1)) = f_1(a_1)$. $\dashv$

Fact 2.5 ([Sh 600] section 4)**.** *Let $\langle M_\alpha : \alpha \leq \theta \rangle$ be an increasing continuous sequence of models in $\mathfrak{k}_\lambda$. Let $N \succ M_0$, and for $\alpha < \theta$, let $a_\alpha \in M_{\alpha+1} - M_\alpha$, $(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^{3,bs}$ and $b \in N - M_0$, $(M_0, N, b) \in K^{3,bs}$. Then for some $f, \langle N_\alpha : \alpha \leq \theta \rangle$ the following hold:*

$$\begin{tikzcd}
N \arrow[r, "f"] & N_1 \arrow[r] & N_2 \arrow[r] & N_\alpha \arrow[r] & N_{\alpha+1} \arrow[r] & N_\theta \\
M_0 \arrow[r] \arrow[u] & M_1 \arrow[r] \arrow[u] & M_2 \arrow[r] \arrow[u] & M_\alpha \arrow[r] \arrow[u] & M_{\alpha+1} \arrow[r] \arrow[u] & M_\theta
\end{tikzcd}$$

(a) f is an isomorphism of N to N_1 over M_0.
(b) $\langle N_\alpha : \alpha \leq \theta \rangle$ is an increasing continuous sequence.
(c) $M_\alpha \preceq N_\alpha$.
(d) $tp(a_\alpha, N_\alpha, N_{\alpha+1})$ does not fork over M_0.
(e) $tp(f(b), M_\alpha, N_\alpha)$ does not fork over M_0.

For completeness, we give a proof:

Proof. First we explain the idea of the proof. Suppose $M_0 \preceq M_1,\ M_0 \preceq M_2$. Then there is an amalgamation M_3, f_1, f_2 of M_1, M_2 over M_0. such that $f_1 = id_{M_1}$. There is also such an amalgamation such that $f_2 = id_{M_2}$. But maybe there is no such an amalgamation such that $f_1 = id_{M_1}$ and $f_2 = id_{M_2}$. So we have to choose if we want to "fix" M_1 or M_2. In our case we have to amalgamate N with another model θ times. So if we want to "fix" the models in the sequence $\langle M_\alpha : \alpha \leq \theta \rangle$, then we will "change" N θ times. So in limit steps we will be in a problem. The solution is to fix N, and "change" the sequence $\langle M_\alpha : \alpha \leq \theta \rangle$. In the end of the proof we "return the sequence to its place".

The proof itself: We choose (N^*_α, f_α) by induction on α such that:

(1) $\alpha \leq \theta \Rightarrow N^*_\alpha \in K_\lambda$.
(2) $(N^*_0, f_0) = (N, id_{M_0})$.
(3) The sequence $\langle N^*_\alpha : \alpha \leq \theta \rangle$ is increasing and continuous.
(4) The sequence $\langle f_\alpha : \alpha \leq \theta \rangle$ is increasing and continuous.
(5) For $\alpha \leq \theta$, the function f_α is an embedding of M_α to N^*_α.
(6) $tp(f_\alpha(a_\alpha), N^*_\alpha, N^*_{\alpha+1})$ does not fork over $f_\alpha[M_\alpha]$.
(7) $tp(b, f_\alpha[M_\alpha], N^*_\alpha)$ does not fork over M_0.

Why is this possible? For $\alpha = 0$ see 2. For α limit define $N^*_\alpha := \bigcup\{N^*_\beta : \beta < \alpha\}$, $f_\alpha := \bigcup\{f_\beta : \beta < \alpha\}$. By the induction hypothesis $\beta < \alpha \Rightarrow f_\beta[M_\beta] \preceq N^*_\beta$ and the sequences $\langle N^*_\beta : \beta \preceq \alpha \rangle$, $\langle f_\beta : \beta \preceq \alpha \rangle$ are increasing and continuous. So by the smoothness (Definition 1.1.d) $f_\alpha[M_\alpha] \preceq N^*_\alpha$. By the induction hypothesis for $\beta \in \alpha$ the type $tp(b, f_\beta[M_\beta], N^*_\beta)$ does not fork over M_0. So by Definition 2.1.h (continuity), the type $tp(b, f_\alpha[M_\alpha], N^*_\alpha)$ does not fork over M_0. Why can we define (N^*_α, f_α) for $\alpha = \beta+1$? Let $f_{\beta+0.5}$ be a function with domain M_α which extend f_β. By condition 5 in the induction hypothesis, $f_\beta[M_\beta] \preceq N^*_\beta$, and obviously $f_\beta[M_\beta] \preceq f_{\beta+0.5}[M_\alpha]$. By assumption, $tp(a_\beta, M_\beta, M_\alpha) \in S^{bs}(M_\beta)$. So $tp(f_{\beta+0.5}(a_\beta), f_\beta[M_\beta], f_{\beta+0.5}[M_\alpha]) \in S^{bs}(f_\beta[M_\beta])$. By condition 7 in the induction hypothesis, $tp(b, f_\beta[M_\beta], N^*_\beta) \in S^{bs}(f_\beta[M_\beta])$. So by Definition 2.1.f, there are a model $N_{\beta+1} \succ N_\beta$ and an embedding $f_{\beta+1} \supseteq f_\beta$ such that condition 6 is satisfied and the type $tp(b, f_{\beta+1}[M_{\beta+1}], N^*_{\beta+1})$ does not fork over $f_\beta[M_\beta]$. By Proposition 2.3 condition 7 is satisfied. So we can choose by induction N^*_α, f_α.

Now $f_\theta : M_\theta \to N^*_\theta$ is an embedding. Extend f^{-1}_θ to a function g with domain N^*_θ and define $f := g \restriction N$. By 2,3 $N \preceq N^*_\theta$. By 2, f is an isomorphism over M_0, so requirement a is satisfied. Define $N_\alpha := g[N^*_\alpha]$. By 5, $f_\alpha[M_\alpha] \preceq N^*_\alpha$, so $M_\alpha \preceq N_\alpha$. So requirement c is satisfied. It is easy to see that 3 implies requirement b and 6,7 implies requirements e,f. $\dashv$

3. Independence

From now until the end of the paper we assume that:

Hypothesis 3.1. $\mathfrak{s}$ is a good $\lambda-$frame minus stability.

Definition 3.2.
(a) $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ is said to be *independent* over M when:
1. $\langle M_\alpha : \alpha \leq \alpha^* \rangle$ is an increasing continuous sequence of models in $\mathfrak{k}_\lambda$.
2. $M \preceq M_0$.
3. For $\alpha < \alpha^*$, the type $tp(a_\alpha, M_\alpha, M_{\alpha+1})$ does not fork over M.

(b) $\langle a_\alpha : \alpha < \alpha^* \rangle$ is said to be *independent* in (M, M_0, M^+) when for some increasing continuous sequence $\langle M_\alpha : 0 < \alpha \leq \alpha^* \rangle$ and a model M^{++} the sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ is independent over M and $M^+ \succeq M^{++} \preceq M_{\alpha^*}$.

(c) The *set* J is said to be *independent* in (M, A, N) when:
There is an independent sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ over M such that:
1. $J = \{a_\alpha : \alpha < \alpha^*\}$.
2. $A \subset M_0$.
3. There is a model N^+ such that $M_{\alpha^*} \preceq N^+$ and $N \preceq N^+$.

(d) J is said to be *finitely independent* in (M, A, N) when: every finite subset of J is independent in (M, A, N).

Definition 3.3. We say that *independence is finitely independence* when for every M, M_0, M^+, J the following hold: J is independent in (M, M_0, M^+) iff J is finitely independent in (M, M_0, M^+).

Proposition 3.4. *If the sequence $\langle a_\alpha : \alpha < \alpha^* \rangle$ is independent in (M, M_0, M_2) and $tp(a, M_0, M_1) \in S^{bs}(M_0)$ then for some amalgamation (id_{M_2}, f, M_3) of M_2, M_1 over M_0 the sequence $\langle a_\alpha : \alpha < \alpha^* \rangle$ is independent in $(M, f[M_1], M_3)$ and $tp(f(a), M_2, M_3)$ does not fork over M_0.*

Proof. By Fact 2.5 $\dashv$

Proposition 3.5. *If $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ is an independent sequence over M and $p \in S^{bs}(M_{\alpha^*})$, then there is a sequence $\langle N_\alpha : \alpha \leq \alpha^* \rangle$ such that:*
1. *For $\alpha \leq \alpha^*$, $M_\alpha \preceq N_\alpha$.*
2. *For $\alpha < \alpha^*$ the type $tp(a_\alpha, N_\alpha, N_{\alpha+1})$ does not fork over M_α.*
3. *The sequence $\langle N_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle N_{\alpha^*} \rangle$ is independent over M.*
4. *p is realized in N_{α^*}.*

Proof. First note that clause 3 follows by clauses 1,2 and the transitivity proposition (Proposition 2.3). By the Definition 2.1.c (local character axiom), there is $\alpha_0 < \alpha^*$ such that p does not fork over M_{α_0}. So there is $N \succ M_{\alpha_0}$ which realizes $p \restriction M_{\alpha_0}$, namely there is an element $a \in N$ such that $tp(a, M_{\alpha_0}, N) = p \restriction M_{\alpha_0}$. By Proposition 2.5 there a sequence $\langle N_\alpha : \alpha \in [\alpha_0, \alpha^*] \rangle$ such that clauses 1,2 hold, $tp(a, M_{\alpha^*}, N_{\alpha^*})$ does not fork over M_{α_0} and N_{α_0}, N are isomorphic over M_{α_0}. By Definition 2.1.d (uniqueness) $tp(a, M_{\alpha^*}, N_{\alpha^*}) = p$. Now define $N_\alpha := N_{\alpha_0}$ for $\alpha < \alpha_0$. $\dashv$

Proposition 3.6. *If $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ is independent over M and $M_{\alpha^*} \prec M^+$, then there is a sequence $\langle N_\alpha : \alpha \leq \alpha^* \rangle$ such that:*

(1) For $\alpha \leq \alpha^*$, $M_\alpha \preceq N_\alpha$.
(2) $N_0 = M_0$.
(3) The sequence $\langle N_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle N_{\alpha^*} \rangle$ is independent over M.
(4) $M^+ \preceq N_{\alpha^*}$.

$$\begin{tikzcd}
N_0 \arrow[r, "id"] & N_1 \arrow[r, "id"] & N_\alpha \arrow[r, "id"] & N_{\alpha+1} \arrow[r, "id"] & N_{\alpha^*} \\
 & & & & M^+ \arrow[u, "id"] \\
M_0 \arrow[r, "id"] \arrow[uu, "id"] & M_1 \arrow[r, "id"] \arrow[uu, "id"] & M_\alpha \arrow[r, "id"] \arrow[uu, "id"] & M_{\alpha+1} \arrow[r, "id"] \arrow[uu, "id"] & M_{\alpha^*} \arrow[u, "id"]
\end{tikzcd}$$

Proof. The idea is to find λ^+ candidates for $\langle N_\alpha : \alpha \leq \alpha^* \rangle$. If none of them satisfies the conclusion, then we get a contradiction.

We try to choose by induction on $\beta < \lambda^+$ a candidate $\langle M_{\beta,\alpha} : \alpha \leq \alpha^* + 1 \rangle$ such that:

(i) $\langle M_{\beta,\alpha} : \alpha \leq \alpha^* + 1 \rangle$ is increasing and continuous.
(ii) For $\alpha \leq \alpha^*$ $M_{0,\alpha} = M_\alpha$.
(iii) $M_{0,\alpha^*+1} := M^+$.
(iv) For $\alpha \leq \alpha^* + 1$, $M_{\beta,\alpha} \preceq M_{\beta+1,\alpha}$.
(v) For $\alpha < \alpha^*$, $tp(a_\alpha, M_{\beta+1,\alpha}, M_{\beta+1,\alpha+1})$ does not fork over $M_{\beta,\alpha}$.
(vi) If β is limit then for $\alpha \leq \alpha^* + 1$ $M_{\beta,\alpha} = \bigcup \{M_{\gamma,\alpha} : \gamma < \beta\}$.
(vii) $M_{\beta+1,\alpha^*} \bigcap M_{\beta,\alpha^*+1} \neq M_{\beta,\alpha^*}$.

$$\begin{tikzcd}
M_{\beta+1,0} \arrow[r, "id"] & M_{\beta+1,\alpha} \arrow[r, "id"] & M_{\beta+1,\alpha+1} \arrow[r, "id"] & M_{\beta+1,\alpha^*} \arrow[r, "id"] & M_{\beta+1,\alpha^*+1} \\
M_{\beta,0} \arrow[r, "id"] \arrow[u, "id"] & M_{\beta,\alpha} \arrow[r, "id"] \arrow[u, "id"] & M_{\beta,\alpha+1} \arrow[r, "id"] \arrow[u, "id"] & M_{\beta,\alpha^*} \arrow[r, "id"] \arrow[u, "id"] & M_{\beta,\alpha^*+1} \arrow[u, "id"] \\
M_{0,0} \arrow[r, "id"] \arrow[u, "id"] & M_{0,\alpha} \arrow[r, "id"] \arrow[u, "id"] & M_{0,\alpha+1} \arrow[r, "id"] \arrow[u, "id"] & M_{0,\alpha^*} \arrow[r, "id"] \arrow[u, "id"] & M_{0,\alpha^*+1} = M^+ \arrow[u, "id"]
\end{tikzcd}$$

We cannot succeed, because if we succeed then letting $M^* := \bigcup \{M_{\beta,\alpha^*} : \beta < \lambda^+\}$, the sequences $\langle M^* \bigcap M_{\beta,\alpha^*+1} : \beta < \lambda^+ \rangle$, $\langle M_{\beta,\alpha^*} : \beta < \lambda^+ \rangle$ are representations of M^*. So for a club of $\beta < \lambda^+$ $M^* \bigcap M_{\beta,\alpha^*+1} = M_{\beta,\alpha^*}$. Take such a β.

$$M_{\beta,\alpha^*} \subseteq M_{\beta+1,\alpha^*} \bigcap M_{\beta,\alpha^*+1} \subseteq M^* \bigcap M_{\beta,\alpha^*+1} = M_{\beta,\alpha^*}$$

, hence this is an equivalences chain, in contradiction to condition (v).

Where will we get stuck? Obviously we will not get stuck at $\beta = 0$. For β limit we define $M_{\beta,\alpha} = \bigcup \{M_{\gamma,\alpha} : \gamma < \beta\}$ and by smoothness (Definition 1.1.d) $\langle M_{\beta,\alpha} : \alpha \leq \alpha^* + 1 \rangle$ is increasing. It remains to get stuck at some

successor ordinal. Let β be the first ordinal such that there is no $\langle M_{\beta+1,\alpha} : \alpha \leq \alpha^*+1\rangle$ which satisfies clauses (i)-(vii).

Case a: $M_{\beta,\alpha^*+1} = M_{\beta,\alpha^*}$. We define $N_\alpha := M_{\beta,\alpha}$ for $\alpha \leq \alpha^*+1$. $M^+ = M_{0,\alpha^*+1} \preceq M_{\beta,\alpha^*+1} = M_{\beta,\alpha^*} = N_{\alpha^*}$. So the proposition is proved.

Case b: $M_{\beta,\alpha^*} \prec M_{\beta,\alpha^*+1}$. We prove that we will not get stuck. By the densite of basic types (in Definition 2.1) there is $a \in M_{\beta,\alpha^*+1}$ such that $(M_{\beta,\alpha^*}, M_{\beta,\alpha^*+1}, a) \in S^{bs}(M_{\beta,\alpha^*})$. Denote $p_\beta := tp(a, M_{\beta,\alpha^*}, M_{\beta,\alpha^*+1})$. By Proposition 3.5 there is a sequence $\langle M_{\beta+1,\alpha} : \alpha \leq \alpha^*\rangle$ which satisfies clauses $(i),(iii)$ such that p is realized in $M_{\beta+1,\alpha^*}$. Take $b \in M_{\beta+1,\alpha^*}$ which realizes p. By the definition of a type without loss of generality $a = b$. $\dashv$

Proposition 3.7.
(a) if the set J is independent in (M,A,N) then there is an independent sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle M_{\alpha^*}\rangle$ over M such that $A \subseteq M_0$, $J = \{a_\alpha : \alpha < \alpha^*\}$ and $N \preceq M_{\alpha^*}$.
(b) if the sequence $\langle a_\alpha : \alpha < \alpha^*\rangle$ is independent in (M,A,N) then there is an increasing continuous sequence $\langle M_\alpha : \alpha \leq \alpha^*\rangle$ such that $A \subseteq M_0$, the sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle M_{\alpha^*}\rangle$ is independent over M and $N \preceq M_{\alpha^*}$.

Proof. By Proposition 3.6. $\dashv$

Proposition 3.8 (2-concatenation)**.** *If*

(1) $M \preceq M_0 \preceq M_1 \preceq M_2$.
(2) $\langle a_{0,\beta} : \beta < \beta_0\rangle$ is independent in (M, M_0, M_1).
(3) $\langle a_{1,\beta} : \beta < \beta_1\rangle$ is independent in (M, M_1, M_2).

Then $\langle a_{0,\beta} : \beta < \beta_0\rangle^\frown\langle a_{1,\beta} : \beta < \beta_1\rangle$ is independent in (M, M_0, M_2).

Proof.

$$\begin{tikzcd}
& & & & M_{0,\beta_0+\beta_1} \\
& & & M_{0,\beta_0+1} \arrow[ur, "id"] & \\
& & M_{0,\beta_0} \arrow[ur, "f"] & & M_{1,\beta_1} \arrow[uu, "id"'] \\
& M_{0,\beta} \arrow[ur, "id"] & & M_{1,1} \arrow[uu, "id"] \arrow[ur, "id"] & \\
M_0 \arrow[ur, "id"] \arrow[rr, "id"] & & M_1 \arrow[uu, "id"] \arrow[ur, "id"] \arrow[rr, "id"] & & M_2 \arrow[uu, "id"]
\end{tikzcd}$$

By clause 2 and Proposition 3.7, there is a sequence $\langle M_{0,\beta} : \beta \leq \beta_0\rangle$ such that the sequence $\langle M_{0,\beta}, a_{0,\beta} : \beta < \beta_0\rangle^\frown\langle M_{0,\beta_0}\rangle$ is independent over M and $M_1 \preceq M_{0,\beta_0}$. Similarly by clause 3 and Proposition 3.7 there is a sequence

$\langle M_{1,\beta} : \beta \leq \beta_1 \rangle$ such that the sequence $\langle M_{1,\beta}, a_{1,\beta} : \beta < \beta_1 \rangle^\frown \langle M_{1,\beta_1} \rangle$ is independent over M and $M_2 \preceq M_{1,\beta_1}$. By Fact 2.5, for some $f, \langle M_{0,\beta_0+\beta} : \beta \leq \beta_1 \rangle$ the following hold:

(a) $f : M_{0,\beta_0} \to M_{0,\beta_0+1}$ is an embedding over M_1.
(b) $\langle M_{0,\beta_0+\beta} : \beta \leq \beta_1 \rangle$ is an increasing continuous sequence.
(c) For $\beta \leq \beta_1$, $M_{1,\beta} \preceq M_{0,\beta_0+\beta}$.
(d) For $\beta < \beta_1$, $tp(a_{1,\beta}, M_{0,\beta_0+\beta}, M_{0,\beta_0+\beta+1})$ does not fork over $M_{1,\beta}$.

Since for $\beta < \beta_0$, $a_{0,\beta} \in M_1$ it follows that $f(a_{0,\beta}) = a_{0,\beta}$. Therefore the sequence $\langle f[M_{0,\beta}], a_{0,\beta} : \beta < \beta_0 \rangle^\frown \langle M_{0,\beta_0+\beta}, a_{1,\beta} : \beta < \beta_1 \rangle^\frown \langle M_{0,\beta_0+\beta_1} \rangle$ is independent over M. But $M_2 \preceq M_{1,\beta_1} \preceq M_{0,\beta_0+\beta_1}$. $\dashv$

Theorem 3.9.
(a) If a finite set J *is independent in* (M, M_0, M^+) *and* $\{a_\alpha : \alpha < \alpha^*\}$ *is an enumeration of* J *without repetitions then the* sequence $\langle a_\alpha : \alpha < \alpha^* \rangle$ *is independent in* (M, M_0, M^+).
(b) If the sequence $\langle M_\alpha, a_\alpha : \alpha \leq \beta \rangle^\frown \langle M_{\beta+1} \rangle$ *is independent over* M*, Then the sequence* $\langle a_\beta \rangle^\frown \langle a_\alpha : \alpha < \beta \rangle$ *is independent in* $(M, M_0, M_{\beta+1})$.
(c) If the sequence $\langle a_\alpha : \alpha < \beta + 1 \rangle$ *is independent in* (M, M_0, N)*, then the sequence* $\langle a_\beta \rangle^\frown \langle a_\alpha : \alpha < \beta \rangle$ *is independent in* (M, M_0, N).
(d) If the sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ *is independent over* M *and* $\beta_0 < \beta_1 < \alpha^*$ *then the sequence* $\langle a_{\beta_1} \rangle^\frown \langle a_\alpha : \alpha \in [\beta_0, \beta_1) \rangle^\frown \langle a_\alpha : \alpha \in (\beta_1, \alpha^*) \rangle$ *is independent in* $(M, M_{\beta_0}, M_{\beta_1+1})$*. So the sequence* $\langle a_\alpha : \alpha < \beta_0 \rangle^\frown \langle a_{\beta_1} \rangle^\frown \langle a_\alpha : \alpha \in [\beta_0, \beta_1) \rangle^\frown \langle a_\alpha : \alpha \in (\beta_1, \alpha^*) \rangle$ *is independent in* (M, M_0, M_{α^*}).

Proof.
(a) Follows by item d.
(b) By Fact 2.5 for some $f, \langle N_\alpha : \alpha \leq \beta \rangle$ the following hold:

(1) f is an isomorphism of $M_{\beta+1}$ to N_1 over M_0.
(2) $N_0 = f[M_{\beta+1}]$.
(3) for $\alpha \leq \beta$, $M_\alpha \preceq N_\alpha$.
(4) $tp(f(a_\beta), M_\beta, N_\beta)$ does not fork over M_0.
(5) For $\alpha < \beta$, $tp(a_\alpha, N_\alpha, N_{\alpha+1})$ does not fork over M_α.

$$\begin{tikzcd}
a_\beta \in M_{\beta+1} \arrow[r, "f"] & N_1 \arrow[r, "id"] & N_\beta \arrow[r, "g"] & N_{\beta+1} \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_1 \arrow[u, "id"] \arrow[r, "id"] & M_\beta \arrow[u, "id"] \arrow[r, "id"] & M_{\beta+1} \ni a_\beta \arrow[u, "id"]
\end{tikzcd}$$

By assumption, $tp(a_\alpha, M_\alpha, M_{\alpha+1})$ does not fork over M. Hence by Proposition 2.3 and clause 5 $tp(a_\alpha, N_\alpha, N_{\alpha+1})$ does not fork over M. Therefore:

(6) The sequence $\langle N_\alpha, a_\alpha : \alpha < \beta \rangle^\frown \langle N_\beta \rangle$ is independent in (M, N_0, N_β).

Since f is an isomorphism over M_0, by clause 2 $tp(f(a_\beta), M_0, N_0) = tp(a_\beta, M_0, M_{\beta+1})$. By assumption, $tp(a_\beta, M_\beta, M_{\beta+1})$ does not fork over M_0. So by clause 4 and Definition 2.1.d (uniqueness) $tp(f(a_\beta), M_\beta, N_\beta) =$

$tp(a_\beta, M_\beta, M_{\beta+1})$. hence there is an amalgamation $(g, id_{M_{\beta+1}}, N_{\beta+1})$ of $N_\beta, M_{\beta+1}$ over M_β such that $g(f(a_\beta)) = a_\beta$. Since g is an embedding over M_β, it follows that $g(a_\alpha) = a_\alpha$. Therefore by clause 6 the sequence $\langle M_0, a_\beta\rangle^\frown\langle g[N_\alpha], a_\alpha : \alpha < \beta\rangle^\frown\langle g[N_\beta]\rangle$ is independent over M.

(c) By item b and Proposition 3.7.

(d) By definition the sequence $\langle a_\alpha : \alpha \in [\beta_0, \beta_1]\rangle$ is independent in $(M, M_{\beta_0}, M_{\beta_1+1})$. So by item c, the sequence $\langle a_{\beta_1}\rangle^\frown\langle a_\alpha : \alpha \in [\beta_0, \beta_1)\rangle$ is independent in $(M, M_{\beta_0}, M_{\beta_1+1})$. Now use Proposition 3.8 twice. $\dashv$

The following proposition is similar to Claim 5.13 of [Sh 705].

Proposition 3.10 (The exchange proposition)**.** *Let J be an independent set in (M, M_0, N).*

(1) If $tp(a, M, N) \in S^{bs}(M)$, then there is a finite subset $J^* \subset J$ such that $J \bigcup \{a\} - J^*$ is independent in (M, M_0, N).

(2) If $a \in N$ then for some models M', N' and a subset $J^* \subseteq J$ the following hold:
- *(a) $|J^*| < \aleph_0$.*
- *(b) $M \bigcup \{a\} \subseteq M' \preceq N'$.*
- *(c) $N \preceq N'$.*
- *(d) $J - J^*$ is independent in (M, M', N').*

Proof. By assumption, for some independent sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^*\rangle^\frown \langle M_{\alpha^*}\rangle$ over M $\{a_\alpha : \alpha < \alpha^*\} = J$ and there is N^+ such that $N \preceq N^+$ and $M_{\alpha^*} \preceq N^+$. We prove the proposition by induction on α^*.

(1) Assume that $\alpha^* = 0$. So $J = \emptyset$ and $\{a\}$ is independent in (M, M_0, N). For $\alpha^* = \beta + 1$ we can subtract a_β. Assume α^* is limit. By Proposition 3.6 without loss of generality $N^+ = M_{\alpha^*}$ so $a \in M_\alpha$ for some $\alpha < \alpha^*$. By the induction hypotheses, there is a finite subset $J^* \subseteq \{a_\gamma : \gamma < \alpha\}$, such that $\{a_\gamma : \gamma < \alpha\} \bigcup \{a\} - J^*$ is independent in (M, M_0, M_α). But $\{a_\gamma : \alpha \leq \gamma < \alpha^*\}$ is independent in $(M, M_\alpha, M_{\alpha^*})$. Therefore by Proposition 3.8 $\{a_\alpha : \alpha < \alpha^*\} \bigcup \{a\} - J^*$ is independent in (M, M_0, M_{α^*}).

(2) For $\alpha^* = 0$ we define $M' := N,\ N' := N,\ J^* := \emptyset$ ($J - J^* = \emptyset$, so clause d is not relevant). For $\alpha^* = \beta + 1$ we can subtract a_β: By the induction hypotheses for some models M', N' and a subset $J^* \subseteq \{a_\alpha : \alpha < \beta\}$ the following hold:

(a) $|J^*| < \aleph_0$.
(b) $M \bigcup \{a\} \subseteq M' \preceq N'$.
(c) $N \preceq N'$.
(d) $\{a_\alpha : \alpha < \beta\} - J^*$ is independent in (M, M', N').

Now $M', N', J^* \bigcup \{a_\beta\}$ are as required.

Assume that α^* is a limit ordinal. By Proposition 3.6 without loss of generality $N = M_{\alpha^*}$ so $a \in M_\alpha$ for some $\alpha < \alpha^*$. So by the induction hypotheses, there are M', N', J^* such that the following hold:

(a) $|J^*| < \aleph_0$.
(b) $M \bigcup \{a\} \subseteq M' \preceq N'$.

(c) $M_\alpha \preceq N'$.
(d) $\{a_\beta : \beta < \alpha\} - J^*$ is independent in (M, M', N').

By Proposition 3.4 we can find an amalgamation $(id_{M_{\alpha^*}}, f, N^+)$ of N', M_{α^*} over M_α such that the set $\{a_\beta : \alpha \leq \beta < \alpha^*\}$ is independent in $(M, f[N'], N^+)$.

$$\begin{tikzcd}
a \in M' \arrow[r, "id"] & N' \arrow[r, "f"] & N^+ \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_\alpha \arrow[u, "id"] \arrow[r, "id"] & M_{\alpha^*} = N \arrow[u, "id"]
\end{tikzcd}$$

Since f is an isomorphism over M_α, $\{a_\beta : \beta < \alpha\} - J^*$ is independent in $(M, f[M'], f[N'])$. Now by Proposition 3.8 $J - J^*$ is independent in $(M, f[M'], N^+)$. Since $a \in M_\alpha$, it follows that $a = f(a) \in f[M']$. Therefore $f[M'], N^+, J^*$ are as required. $\dashv$

Definition 3.11. We say that *the independence relation has continuity* when the following holds: If

(1) δ is a limit ordinal.
(2) $\langle M_\alpha : \alpha \leq \delta \rangle$ is an increasing continuous sequence.
(3) For $\alpha < \delta$ the set J is independent in (M, M_α, M^+).

Then J is independent in (M, M_δ, M^+).

Definition 3.12. We say that *the finite independence relation has continuity* when the following holds: If

(1) δ is a limit ordinal.
(2) $\langle M_\alpha : \alpha \leq \delta \rangle$ is an increasing continuous sequence.
(3) For $\alpha < \delta$ the set J is finitely independent in (M, M_α, M^+).

Then J is finitely independent in (M, M_δ, M^+).

Proposition 3.13. *Assume:*

(1) $P \subseteq S^{bs}(M_0)$.
(2) J_1, J_2 *are maximal sets in* $\{J : J$ *is independent in* (M, M_0, N) *and* $a \in J \Rightarrow tp(a, M_0, N) \in P\}$
(3) The independence relation has continuity or J_1 *is finite or* J_2 *is finite.*

Then $|J_1| = |J_2|$ *or they both finite.*

Proof. Towards a contradiction assume that $|J_1| < |J_2| \geq \aleph_0$. Let $\{a_\alpha : \alpha < |J_1|\}$ be an enumeration of J_1 without repetitions. We define $\mu := |J_1|$. We choose by induction on $\alpha \leq \mu$ a triple $(M'_\alpha, N'_\alpha, J^*_\alpha)$ such that:

(1) $\langle M'_\alpha : \alpha \leq |J_1| \rangle$ is an increasing continuous sequence of models.
(2) $\langle N'_\alpha : \alpha \leq |J_1| \rangle$ is an increasing continuous sequence of models.
(3) For $\alpha \leq |J_1|$, $M'_\alpha \preceq N'_\alpha$.
(4) $M'_0 = M_0$.
(5) $N'_0 = N$.

(6) $a_\alpha \in M'_{\alpha+1}$.
(7) J^*_α is a finite subset of $J_2 - \bigcup\{J^*_\beta : \beta < \alpha\}$.
(8) $J_2 - \bigcup\{J^*_\beta : \beta \leq \alpha\}$ is independent in $(M, M'_\alpha, N'_\alpha)$.

Why can we carry out this induction? For $\alpha = 0$ we choose $M'_0 := M_0, N'_0 = N, J^*_0 := \emptyset$.

In the $\alpha + 1$ step, we use Proposition 3.10.2 (the exchange proposition). How? We substitute $J_2 - \{J^*_\beta : \beta \leq \alpha\}, M, M'_\alpha, N'_\alpha, a_\alpha$ instead of J, M, M_0, N, a. By clause 8, $J_2 - \{J^*_\beta : \beta \leq \alpha\}$ is independent in $(M, M'_\alpha, N'_\alpha)$. $a_\alpha \in J_1$, so by assumption 2, $a_\alpha \in N$. By clauses 2,5 $N \subseteq N'_\alpha$, so $a_\alpha \in N'_\alpha$. Therefore by the exchange proposition, for some M', N' and a subset $J^* \subseteq J_2 - \bigcup\{J^*_\beta : \beta \leq \alpha\}$ the following hold:

(1) $|J^*| < \aleph_0$.
(2) $M'_\alpha \bigcup\{a_\alpha\} \subseteq M' \preceq N'$.
(3) $N'_\alpha \preceq N'$.
(4) $J_2 - \bigcup\{J^*_\beta : \beta \leq \alpha\} - J^*$ is independent in (M, M', N').

Now we define $M'_{\alpha+1} := M',\ N'_{\alpha+1} := N',\ J^*_{\alpha+1} := J^*$.

For limit α we define $M'_\alpha := \bigcup\{M'_\beta : \beta < \alpha\},\ N'_\alpha := \bigcup\{N'_\beta : \beta < \alpha\}$ and $J^*_\alpha := \emptyset$. For every $\beta < \alpha$ by clause 8 the set $J_2 - \bigcup\{J^*_\gamma : \gamma \leq \beta\}$ is independent in (M, M'_β, N'_β). So $J_2 - \bigcup\{J^*_\beta : \beta < \alpha\}$ is independent in (M, M'_β, N'_β). Hence by the continuity $J_2 - \bigcup\{J^*_\beta : \beta < \alpha\}$ is independent in $(M, M'_\alpha, N'_\alpha)$.

By assumption, J_1 is independent in (M, M_0, N). By clauses 6,1 $J_1 \subseteq M'_\mu$. But $M'_\mu \preceq N'_\mu \succeq N_0$. Hence J_1 is independent in (M, M_0, M'_μ). By clause 8 $J_2 - \bigcup\{J^*_\beta : \beta \leq \mu\}$ is independent in (M, M'_μ, N'_μ). Therefore by Proposition 3.8 $J_1 \bigcup(J_2 - \bigcup\{J^*_\beta : \beta \leq \mu\})$ is independent in (M, M_0, N'_μ), hence in (M, M_0, N). But it contradicts the maximality of J_1! $(J_1 \bigcup J_2 - \bigcup\{J^*_\beta : \beta \leq \mu\} \neq J_1$, because $|J_1| < |J_2| > \bigcup\{J^*_\beta : \beta \leq \mu\})$. ⊣

Definition 3.14. Suppose $M \preceq_{\mathfrak{k}_\lambda} N$ and let $P \subseteq S^{bs}(M)$.

$$dim(P, N) := Min \left\{ \begin{array}{c|l} |J| & a \in J \Rightarrow tp(a, M, N) \in P \\ & J \text{ is independent in } (M, N) \\ & J \text{ is maximal under the previous conditions} \end{array} \right\}.$$

For $p \in S^{bs}(M)$ we define $dim(p, N) = dim(\{p\}, N)$.

4. Finite Character

In this section we use uniqueness triples. In Definition 4.3=4.3 we define uniqueness triples. In Definition 4.8=4.4 we define an independent sequence by $K^{3,uq}$, the class of uniqueness triples. Proposition 4.10=4.9.c asserts that the existence property for $K^{3,uq}$ implies that any independent sequence is independent by $K^{3,uq}$. Proposition **??**=4.10 explain the advantage of uniqueness triples. Proposition **??**=4.11 is similar to Proposition **??**=4.10, but we replace here independent sequence by independent set. Theorem

??=4.12 is the main theorem of the paper. It asserts that independence satisfies the expected properties.

Definition 4.1. Suppose

(1) $M_0 \preceq_{\mathfrak{s}} M_1 \wedge M_0 \preceq_{\mathfrak{s}} M_2$.
(2) For $x = a, b$, (f_1^x, f_2^x, M_3^x) is an amalgamation of M_1, M_2 over M_0.

$(f_1^a, f_2^a, M_3^a), (f_1^b, f_2^b, M_3^b)$ are said to be *equivalent* over M_0 if there are f^a, f^b, M_3 such that the following diagram commutes:

$$\begin{tikzcd}
 & M_3^b \arrow[r, "f^b"] & M_3 \\
M_1 \arrow[ur, "f_1^b"] \arrow[rr, "f_1^a"] & & M_3^a \arrow[u, "f^a"'] \\
M_0 \arrow[u, "id_{M_0}"] \arrow[r, "id_{M_0}"'] & M_2 \arrow[uu, "f_2^b"] \arrow[ur, "f_2^a"'] &
\end{tikzcd}$$

We denote the relation "to be equivalent over M_0" between amalgamations over M_0, by E_{M_0}.

Proposition 4.2 ([JrSh 875])**.** *The relation E_{M_0} is an equivalence relation.*

We will not give here a proof, as we do not use this proposition.

Definition 4.3. $K^{3,uq}$ is the set of triples $(M_{0,0}, M_{1,0}, a) \in S^{bs}(M_{0,0})$ such that for every model $M_{0,1} \succ M_{0,0}$ there is a unique amalgamation (up to $E_{M_{0,0}}$) $(M_{1,1}, f_{1,0}, f_{0,1})$ of $M_{1,0}, M_{0,1}$ over $M_{0,0}$, such that $f_{1,0}(tp(a, M_{1,0}, M_{0,0}))$ does not fork over $M_{0,0}$. A *uniqueness triple* is a triple in $\mathfrak{k}^{3,uq}$.

It is reasonable to assume that the existence property is satisfied by $K^{3,uq}$, because if $N \in K_\lambda$, $|S(N)| \leq \lambda^+$, and the existence property is not satisfied by $K^{3,uq}$, then by the last corollary in section 4 of [JrSh 875] there are $2^{\lambda^{+2}}$ non isomorphic models in $K_{\lambda^{+2}}$ assuming weak set-theoretic assumptions. Note that by Claims 1.18,1.20 of [Sh E46] we have the following fact:

Fact 4.4. *Assume $\lambda \geq \aleph_0$ and $\mathfrak{k}_\lambda$ has amalgamation. If $N \in K_\lambda$ and $|S(N)| > \lambda^+$, then $|\{N' \in K_\lambda : N \prec N'\}/\cong| > \lambda^+$.*

Remark 4.5. $K^{3,uq}$ is closed under isomorphisms: If $(M_0, M_1, a) \in \mathfrak{k}^{3,uq}$ and $f : M_1 \to M_1^*$ is an isomorphism, then $(f[M_0], f[M_1], f(a)) \in \mathfrak{k}^{3,uq}$.

Proposition 4.6. *If*

(1) $M_{0,0} \preceq M_{1,0} \preceq M_{1,1}$.
(2) $M_{0,0} \preceq M_{0,1} \preceq M_{1,1}$.
(3) $(M_{0,0}, M_{1,0}, a) \in K^{3,uq}$.
(4) $tp(a, M_{0,1}, M_{1,1})$ *does not fork over* $M_{0,0}$.
(5) $tp(b, M_{0,0}, M_{0,1}) \in S^{bs}(M_{0,0})$.

Then $tp(b, M_{1,0}, M_{1,1})$ *does not fork over* $M_{0,0}$.

Proof. We have two amalgamations of $M_{1,0}, M_{0,1}$ over $M_{0,0}$ such that the image of $tp(a, M_{0,1}, M_{1,1})$ does not fork over $M_{0,0}$: One is the amalgamation $(M_{1,1}, id_{M_{1,0}}, id_{M_{0,1}})$. The second exists by Fact 2.4. So by Definition 4.3 they are equivalent. So as in the second amalgamation the types do not fork, so does in the first. ⊣

Proposition 4.7. *If* $(M_0, M_1, a) \in \mathfrak{k}^{3,uq}$, $(M_0, M_2, a) \in \mathfrak{k}^{3,bs}$ *and* $M_0 \preceq M_2 \preceq M_1$ *then* $(M_0, M_2, a) \in \mathfrak{k}^{3,uq}$.

Proof. Easy. ⊣

Definition 4.8. $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ is said to be independent over M *by uniqueness triples* when we add in Definition 3.2: $(M_\alpha, M_{\alpha+1}, a_\alpha) \in K^{3,uq}$. Similarly for independent set.

Roughly speaking the following proposition says that: Suppose that we have an independent sequence.

(a) One can replace the first triple in the sequence by any other triple with the same type.
(b) Like item 1, but for all the triples simultaneously, i.e. if someone chooses the triples (up to isomorphisms), we will still be able to find a witness for independence.
(c) If the existence property is satisfied by $K^{3,uq}$, then the sequence is independent by uniqueness triples.

Definition 4.9. Suppose $\langle M_{0,\alpha}, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{0,\alpha^*} \rangle$ is an independent sequence over M. *The independence game* for the sequence $\langle M_{0,\alpha}, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{0,\alpha^*} \rangle$ over M is a two-player game that lasts $\alpha^* + 1$ moves.

$$\begin{tikzcd}
b_2 \in M_{3,0} \arrow[rrr, "f_3"] & & & M_{3,3} \\
b_1 \in M_{2,0} \arrow[u, "id"] \arrow[rr, "f_2"] & & M_{2,2} \arrow[r, "id"] & M_{2,3} \arrow[u, "id"'] \\
b_0 \in M_{1,0} \arrow[u, "id"] \arrow[r, "f_1"] & M_{1,1} \arrow[r, "id"] & M_{1,2} \arrow[u, "id"] \arrow[r, "id"] & M_{1,3} \arrow[u, "id"'] \\
M_{0,0} \arrow[u, "id"] \arrow[r, "f_0"] & a_0 \in M_{0,1} \arrow[u, "id"] \arrow[r, "id"] & a_1 \in M_{0,2} \arrow[u, "id"] \arrow[r, "id"] & a_2 \in M_{0,3} \arrow[u, "id"']
\end{tikzcd}$$

The $\alpha+1$ move: Player 1 chooses $M_{\alpha+1,0}, b_\alpha$, such that $tp(b_\alpha, M_{\alpha,0}, M_{\alpha+1,0}) = tp(a_\alpha, M_{\alpha,0}, M_{\alpha,\alpha+1})$. Then player 2 chooses $f_{\alpha+1}$ and a sequence $\langle M_{\alpha+1,\beta} : \beta \in [\alpha+1, \alpha^*] \rangle$ (for $\beta \in (0, \alpha+1)$, $M_{\alpha+1,\beta}$ is not defined) such that:

(1) $f_{\alpha+1}$ is an injection with domain $M_{\alpha+1,0}$.
(2) $f_\alpha \subset f_{\alpha+1}$.
(3) $f_{\alpha+1}(b_\alpha) = a_\alpha$.
(4) For $\beta \in (\alpha, \alpha^*)$ $tp(a_\beta, M_{\alpha+1,\beta}, M_{\alpha+1,\beta+1})$ does not fork over $M_{\alpha,\beta}$.

For $\alpha = 0$ we define $f_0 := id_{M_{0,0}}$. For α limit, in the α move, we define $M_{\alpha,0} := \bigcup\{M_{\beta,0} : \gamma < \alpha\}$, $f_\alpha := \bigcup\{f_\gamma : \gamma < \alpha\}$ and for $\beta \in [\alpha, \alpha^*]$ $M_{\alpha,\beta} := \bigcup\{M_{\gamma,\beta} : \gamma < \alpha\}$. So for $\alpha = 0$ or limit, in the α move the players do not have any choice. Player 2 wins if he has always a legal move (so in this case, in the end of the game, the sequence $\langle M_{\alpha,\alpha}, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle M_{\alpha^*,\alpha^*}\rangle$ is independent over M).

Proposition 4.10. *Suppose: The sequence $\langle M_{0,\alpha}, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle M_{0,\alpha^*}\rangle$ is independent over M.*

(a) If $tp(b, M_{0,0}, M_{1,0}) = tp(a_0, M_{0,0}, M_{0,1})$, Then for some sequence $\langle M_{1,\alpha} : 0 < \alpha \leq \alpha^*\rangle$ and f the following hold:

1. *$\langle M_{1,\alpha} : \alpha \leq \alpha^*\rangle$ is an increasing continuous sequence of models in K_λ.*
2. *$M_{0,\alpha} \preceq M_{1,\alpha}$ for each $\alpha \leq \alpha^*$.*
3. *$0 < \alpha < \alpha^* \Rightarrow a_\alpha \in M_{1,\alpha+1} - M_{1,\alpha}$.*
4. *$0 < \alpha < \alpha^* \Rightarrow tp(a_\alpha, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over M.*
5. *$f : M_{1,0} \to M_{1,1}$ is an embedding over $M_{0,0}$, and $f(b) = a_0$.*
6. *The sequence $\langle M_{0,0}, a_0, f[M_{1,0}], a_1\rangle^\frown\langle M_{1,\alpha}, a_\alpha : 1 < \alpha < \alpha^*\rangle^\frown\langle M_{1,\alpha^*}\rangle$ is independent over M.*

$$\begin{tikzcd}
b \in M_{1,0} \arrow[r, "f"] & M_{1,1} \arrow[r] & a_1 \in M_{1,2} \arrow[r] & a_2 \in M_{1,3} \\
M_{0,0} \arrow[u] \arrow[r] & a_0 \in M_{0,1} \arrow[u] \arrow[r] & a_1 \in M_{0,2} \arrow[u] \arrow[r] & a_2 \in M_{0,3} \arrow[u]
\end{tikzcd}$$

(b) Player 2 has a winning strategy in the independence game.

(c) If the existence property is satisfied by $K^{3,uq}$, then the sequence $\langle a_\alpha : \alpha < \alpha^*\rangle$ is independent in (M, M_0, M_{α^*}) by uniqueness triples.

Proof. (a): As $tp(b, M_{0,0}, M_{1,0}) = tp(a_0, M_{0,0}, M_{0,1})$, there are $g, M_{1,1}^{temp}$ such that $M_{0,1} \preceq M_{1,1}^{temp}$, $g : M_{1,0} \to M_{1,1}^{temp}$ is an embedding over $M_{0,0}$, and $g(b) = a_0$. Now by Fact 2.5 (amalgamation of a model and a sequence), for some $h, \langle M_{1,\alpha} : 2 \leq \alpha \leq \alpha^*\rangle$ the following hold:

1. For $\alpha \in [2, \alpha^*]$ we have $M_{0,\alpha} \preceq M_{1,\alpha}$.
2. For $\alpha \in [1, \alpha^*)$ $tp(a_\alpha, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over $M_{0,\alpha}$.
3. $h : M_{1,1}^{temp} \to M_{1,2}$ is an embedding over $M_{0,1}$.

Now define $f := h \circ g$. Clearly $f(b) = a_0$ and f fixes $M_{0,0}$. Why is condition 4 satisfies? complete... Why is condition 5 satisfies? complete...

$(a) \Rightarrow (b)$: By item a, player 2 has always a legal move. Now we have to prove that if player 2 wins the game, then (in the end of the game) the sequence $\langle M_{\beta,\beta}, a_\beta : \beta < \alpha^*\rangle^\frown\langle M_{\alpha^*,\alpha^*}\rangle$ is independent over M.

First, why does the sequence $\langle M_{\beta,\beta} : \beta < \alpha^*\rangle$ is continuous? Take $\beta < \alpha^*$ limit and take an element $x \in M_{\beta,\beta}$. There is an $\alpha < \beta$ such that $x \in M_{\alpha,\beta}$. But $M_{\alpha,\beta} = \bigcup\{M_{\alpha,\varepsilon} : \varepsilon < \beta\}$. so there is an $\varepsilon < \beta$ such that $x \in M_{\alpha,\varepsilon}$. Define $\gamma := Max\{\alpha, \varepsilon\}$. So $x \in M_{\gamma,\gamma}$ and $\gamma < \beta$.

It remains to prove that for $\beta < \alpha^*$ the type $tp(a_\beta, M_{\beta,\beta}, M_{\beta+1,\beta+1})$ does not fork over M. Let $\beta < \alpha^*$. For $\alpha < \beta$, by clause 4 $tp(a_\beta, M_{\alpha+1,\beta}, M_{\alpha+1,\beta+1})$ does not fork over $M_{\alpha,\beta}$. The sequence $\langle M_{\alpha,\beta} : \alpha \leq \beta \rangle$ is increasing and continuous. So $tp(a_\beta, M_{\beta,\beta}, M_{\beta,\beta+1})$ does not fork over $M_{0,\beta}$. But by assumption, the sequence $\langle M_{0,\alpha} : \alpha < \alpha^* \rangle^\frown \langle M_{0,\alpha^*} \rangle$ is independent over M. So $tp(a_\beta, M_{0,\beta}, M_{0,\beta+1})$ does not fork over M. Hence by the transitivity (Proposition 2.3), $tp(a_\beta, M_{\beta,\beta}, M_{\beta+1,\beta})$ does not fork over M. But $tp(a_\beta, M_{\beta,\beta}, M_{\beta+1,\beta+1}) = tp(a_\beta, M_{\beta,\beta}, M_{\beta,\beta+1})$.

$b \Rightarrow c$: In the $\alpha + 1$ step player 1 chooses a triple in $K^{3,uq}$ and player 2 plays a winning strategy. $\dashv$

Proposition 4.11. *If*

(1) $\langle M_\alpha, a_\alpha : \alpha \leq \beta \rangle^\frown \langle M_{\beta+1} \rangle$ *is independent over* M.
(2) $(M_0, N, a_\beta) \in K^{3,uq}$.
(3) $N \preceq M_{\beta+1}$.

Then $\langle a_\alpha : \alpha < \beta \rangle$ *is independent in* $(M, N, M_{\beta+1})$.

Proof. By Proposition 3.4 there is an amalgamation (f, id_{M_β}, M^+) of N, M_β over M_0 such that $tp(f(a_\beta), M_\beta, M^+)$ does not fork over M_0 and $\langle a_\alpha : \alpha < \beta \rangle$ is independent in $(M, f[N], M^+)$. Since $tp(a_\beta, M_\beta, M_{\beta+1})$ does not fork over M_0, by the definition of $K^{3,uq}$, $(f, id_{M_\beta}, M^+) E_{M_0} (id_{M_1}, id_{M_\beta}, M_{\beta+1})$. So for some g, M^{++} the following hold:

$$\begin{tikzcd}
& M^+ \arrow[r, "g"] & M^{++} \\
N \arrow[ur, "f"] \arrow[rr, "id"] & & M_{\beta+1} \arrow[u, "id"'] \\
M_0 \arrow[u, "id"] \arrow[r, "id"] & M_\beta \arrow[uu, "id"] \arrow[ur, "id"] &
\end{tikzcd}$$

(1) $M_{\beta+1} \preceq M^{++}$.
(2) $g : M^+ \to M^{++}$ is an embedding over M_β.
(3) $g \circ f = id_N$.

So $\langle a_\alpha : \alpha < \beta \rangle$ is independent in $(M, g[f[M_1]], g[M^+])$. Therefore it is independent in $(M, N, M_{\beta+1})$.

$\dashv$

Proposition 4.12. *If*

(1) J *is finitely independent in* (M, M_0, M^+).
(2) $a \in J$.
(3) $(M_0, M_1, a) \in K^{3,uq}$.
(4) $M_1 \preceq M^+$.

Then $J - \{a\}$ *is finitely independent in* (M, M_1, M^+).

Proof. let $J^* := \{b_0, b_1, ..., b_{n-1}\}$ be a finite subset of $J - \{a\}$. By assumption, $J^* \bigcup\{a\}$ is independent in (M, M_0, M^+). By Theorem 3.9.a the sequence $\langle b_0, b_1...b_{n-1}\rangle^\frown\langle a\rangle$ is independent in (M, M_0, M^+). So by Proposition 4.11 $\langle b_0, b_1...b_{n-1}\rangle$ is independent in (M, M_1, M^+). $\dashv$

The main theorem:

Theorem 4.13. *Suppose $\mathfrak{s}$ is a good λ-frame minus stability with conjugation and the existence property is satisfied by $K^{3,uq}$.*

(a) Independence is finitely independence.
(b) If J is independent in (M, M_0, N) and the set $\{a_\alpha : \alpha < \alpha^*\}$ is an enumeration of J without repetitions, then the sequence $\langle a_\alpha : \alpha < \alpha^*\rangle$ is independent in (M, M_0, N).
(c) The independence relation has continuity.
(d) If
 (a) $M \preceq N$.
 (b) $P \subset S^{bs}(M)$.
 (c) $J \subset N$.
 (d) $a \in J \Rightarrow tp(a, M, N) \in P$.
 (e) J is independent in (M, M, N).
 (f) J is maximal under the previous conditions.
 Then $dim(P, N) = |J|$ or $dim(P, N) + |J| < \aleph_0$.

Proof. $(a \wedge c) \Rightarrow d$: By Proposition 3.13.
$a \Rightarrow c$: By Proposition 4.19 below.
Therefore it is enough to prove items a,b.

Proposition 4.14. *Items a,b of Theorem 4.13 is implied by (*): If*
 (1) J is finitely independent in (M, M_0, M^+).
 (2) $\{a_\alpha : \alpha < \alpha^*\} \subset J \subseteq M^+$.
 (3) The sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle M_{\alpha^*}\rangle$ is independent over M by $K^{3,uq}$.
 (4) $M_{\alpha^*} \preceq M^+$.
Then $J - \{a_\alpha : \alpha < \alpha^*\}$ is finitely independent in (M, M_{α^*}, M^+).

Proof. (a) We prove the non-trivial direction. Assume that J is finitely independent in (M, M_0, M^+). Denote $N_0 := M^+$. Take an enumeration $\{a_\alpha : \alpha < |J|\}$ of J without repetitions. We choose by induction on $\gamma \in (0, |J|]$ a pair of models (M_γ, N_γ) such that:
* The sequence $\langle M_\alpha, a_\alpha : \alpha < \gamma\rangle^\frown\langle M_\gamma\rangle$ is independent over M by $K^{3,uq}$.
* $\langle N_\alpha : \alpha \leq \gamma\rangle$ is increasing and continuous.
* For $\alpha \leq \gamma$, $M_\alpha \preceq N_\alpha$.

If we succeed to carry out this induction, then the sequence $\langle M_\alpha, a_\alpha : \alpha < |J|\rangle$ is independent over M, so the set J is independent in $(M, M_0, M_{|J|})$ so in (M, M_0, M^+) (because $M^+ \preceq N_{|J|} \succeq M_{|J|}$).

Why can we carry out this induction? Assume that $\gamma = \alpha^* + 1$. We want to substitute N_{α^*} instead of M^+ in assumption $(*)$ and to conclude that the

set $J - \{a_\alpha : \alpha < \alpha^*\}$ is finitely independent in $(M, M_{\alpha^*}, N_{\alpha^*})$. But why are the conditions of $(*)$ satisfied?

(1) Since $N = M^+$.
(2) By the definition of $\{a_\alpha : \alpha < |J|\}$.
(3) By the induction hypothesis.
(4) By the induction hypothesis $M_{\alpha^*} \preceq N_{\alpha^*}$.

Therefore we can conclude that the set $J - \{a_\alpha : \alpha < \alpha^*\}$ is finitely independent in $(M, M_{\alpha^*}, N_{\alpha^*})$.

$\{a_{\alpha^*}\} \subseteq J - \{a_\alpha : \alpha < \alpha^*\}$ and it is finite. So the set $\{a_{\alpha^*}\}$ is independent in $(M, M_{\alpha^*}, N_{\alpha^*})$, namely $(M_{\alpha^*}, N_{\alpha^*}, a_{\alpha^*}) \in K^{3,bs}$ and $tp(a_{\alpha^*}, M_{\alpha^*}, N_{\alpha^*})$ does not fork over M. Since the existence property is satisfied by $K^{3,uq}$, there are $M^{temp}_{\alpha^*+1}, b_{\alpha^*}$ such that $(M_{\alpha^*}, M^{temp}_{\alpha^*+1}, b_{\alpha^*}) \in K^{3,uq}$ and $tp(b_{\alpha^*}, M_{\alpha^*}, M^{temp}_{\alpha^*+1}) = tp(a_{\alpha^*}, M_{\alpha^*}, N_{\alpha^*})$. So by the definition of a type (and Remark 4.5), for some models $M_{\alpha^*+1}, N_{\alpha^*+1}$ the following hold:

* $M_{\alpha^*} \preceq M_{\alpha^*+1} \preceq N_{\alpha^*+1}$.
* $N_{\alpha^*} \preceq N_{\alpha^*+1}$.
* $(M_{\alpha^*}, M_{\alpha^*+1}, a_{\alpha^*}) \in K^{3,uq}$.

For limit γ we take unions and use smoothness.

(b) By a similar proof. $\dashv$

By [JrSh 875] (Definition 5.2 and Theorem 5.27):

Fact 4.15. *Suppose $\mathfrak{s}$ is a good λ-frame minus stability with conjugation and the existence property is satisfied by $K^{3,uq}$. Then there is a (unique) relation $NF \subseteq {}^4K_\lambda$ such that:*

(a) If $NF(M_0, M_1, M_2, M_3)$ then $n \in \{1,2\} \rightarrow M_0 \leq M_n \leq M_3$ and $M_1 \cap M_2 = M_0$.
(b) Monotonicity: if $NF(M_0, M_1, M_2, M_3)$ and $N_0 = M_0, n < 3 \rightarrow N_n \leq M_n \wedge N_0 \leq N_n \leq N_3, (\exists N^*)[M_3 \leq N^* \wedge N_3 \leq N^*]$ then $NF(N_0, N_1, N_2, N_3)$.
(c) Existence: For every $N_0, N_1, N_2 \in K_\lambda$ if $l \in \{1,2\} \rightarrow N_0 \leq N_l$ and $N_1 \bigcap N_2 = N_0$ then there isN_3 s.t. $NF(N_0, N_1, N_2, N_3)$.
(d) Uniqueness: Suppose for x=a,b $NF(N_0, N_1, N_2, N_3^x)$. Then there is a joint embedding of N^a, N^b over $N_1 \bigcup N_2$.
(e) Symmetry: $NF(N_0, N_1, N_2, N_3) \leftrightarrow NF(N_0, N_2, N_1, N_3)$.
(f) Long transitivity: For $x = a, b$ let $\langle M_{x,i} : i \leq \alpha^* \rangle$ an increasing continuous sequence of models in K_λ. Suppose $i < \alpha^* \rightarrow NF(M_{a,i}, M_{a,i+1}, M_{b,i}, M_{b,i+1})$. Then $NF(M_{a,0}, M_{a,\alpha^*}, M_{b,0}, M_{b,\alpha^*})$
(g) NF respects $\mathfrak{s}$: if $NF(M_0, M_1, M_2, M_3)$ and $tp(a, M_0, M_1) \in S^{bs}(M_0)$ then $tp(a, M_2, M_3)$ does not fork over M_0.

Proposition 4.16. *If $\langle M_\alpha : \alpha \leq \alpha^* \rangle$ is an increasing continuous sequence of models and $M_0 \prec N$, then there is an increasing continuous sequence $\langle N_\alpha : \alpha \leq \alpha^* \rangle$ such that for $\alpha < \alpha^*$ $NF(M_\alpha, M_{\alpha+1}, N_\alpha, N_{\alpha+1})$ and N, N_0 are isomorphic over M_0.*

Proof. We choose $(N^{temp}_\alpha, f_\alpha)$ by induction on α such that:

(1) $\alpha \leq \theta \Rightarrow N^{temp}_\alpha \in K_\lambda$.
(2) $(N^{temp}_0, f_0) = (N, id_{M_0})$.
(3) The sequence $\langle N^{temp}_\alpha : \alpha \leq \theta \rangle$ is increasing and continuous.
(4) The sequence $\langle f_\alpha : \alpha \leq \theta \rangle$ is increasing and continuous.
(5) For $\alpha \leq \theta$, the function f_α is an embedding of M_α to N^{temp}_α.
(6) For $\alpha < \theta$, we have $NF(M_\alpha, M_{\alpha+1}, N^{temp}_\alpha, N^{temp}_{\alpha+1})$.

$$\begin{tikzcd}
N = N^{temp}_0 \arrow[r, "id"] & N^{temp}_2 \arrow[r, "id"] & N^{temp}_\alpha \arrow[r, "id"] & N^{temp}_{\alpha+1} \arrow[r, "id"] & N^{temp}_\theta \\
M_0 \arrow[u, "f_0"] \arrow[r, "id"] & M_2 \arrow[u, "f_2"] \arrow[r, "id"] & M_\alpha \arrow[u, "f_\alpha"] \arrow[r, "id"] & M_{\alpha+1} \arrow[u, "f_{\alpha+1}"] \arrow[r, "id"] & M_\theta \arrow[u, "f_\theta"]
\end{tikzcd}$$

Why is this possible? For $\alpha = 0$ see 2. For α limit define $N^{temp}_\alpha := \bigcup\{N^{temp}_\beta : \beta < \alpha\}$, $f_\alpha := \bigcup\{f_\beta : \beta < \alpha\}$. By the induction hypothesis $\beta < \alpha \Rightarrow f_\beta[M_\beta] \preceq N^{temp}_\beta$ and the sequences $\langle N^{temp}_\beta : \beta \leq \alpha\rangle$, $\langle f_\beta : \beta \leq \alpha\rangle$ are increasing and continuous. So by smoothness (Definition 1.1.d) $f_\alpha[M_\alpha] \preceq N^{temp}_\alpha$.

For successor α we use the existence (clause c) in Fact 4.15.

Now $f_\theta : M_\theta \to N^{temp}_\theta$ is an isomorphism. Extend f^{-1}_θ to a function g with domain N^{temp}_θ and define $f := g \restriction N$. By 2,3 $N \preceq N^{temp}_\theta$. By 2, f is an isomorphism over M_0. Define $N_\alpha := g[N^{temp}_\alpha]$. By 5, $f_\alpha[M_\alpha] \preceq N^{temp}_\alpha$, so $M_\alpha \preceq N_\alpha$. ⊣

Proposition 4.17. *If J is independent in (M, M_0, M_1) and $NF(M_0, M_1, M_2, M_3)$ then J is independent in (M, M_2, M_3).*

Proof. By Proposition 4.16, the long transitivity in Fact 4.15 and the uniqueness in Fact 4.15. We elaborate: By Definition 3.2 there is an independent sequence $\langle N_{0,\alpha}, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle N_{0,\alpha^*}\rangle$ over M such that $J = \{a_\alpha : \alpha < \alpha^*\}$, $N_{0,0} = M_0$ and $M_1 \preceq N_{0,\alpha^*}$. By Proposition 4.16 there is an increasing continuous sequence of models $\langle N_{1,\alpha} : \alpha \leq \alpha^*\rangle$ such that for $\alpha \leq \alpha^*$ we have $NF(N_{0,\alpha}, N_{0,\alpha+1}, N_{1,\alpha}, N_{1,\alpha+1})$ and there is an isomorphism $f : M_2 \to N_{1,0}$ over M_0. Now we prove:

(i) $NF(M_0, M_1, N_{1,0}, N_{1,\alpha^*})$.
(ii) The sequence $\langle N_{1,\alpha}, a_\alpha : \alpha < \alpha^*\rangle^\frown\langle N_{1,\alpha^*}\rangle$ is independent over M.

Why is it enough? By assumption $NF(M_0, M_1, M_2, M_3)$, so by (i) and the uniqueness in Fact 4.15 $(id_{M_1}, id_{M_2}, M_3)E_{M_0}(id_{M_1}, f, M^*_3)$. Therefore by (ii), the set J is independent in (M, M_2, M_3).

Proof.

(i) By the long transitivity in Fact 4.15 we have $NF(M_0, N_{0,\alpha^*}, N_{1,0}, N_{1,\alpha^*})$. But $M_0 \preceq M_1 \preceq N_{0,\alpha^*}$. So by the monotonicity in Fact 4.15 $NF(M_0, M_1, N_{1,0}, N_{1,\alpha^*})$.

(ii) By Fact 4.15 the relation NF respects the frame $\mathfrak{s}$. But for $\alpha < \alpha^*$ $NF(M_{0,\alpha}, M_{0,\alpha+1}, M_{1,\alpha}, M_{1,\alpha+1})$ holds. So $tp(a_\alpha, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over $M_{0,\alpha}$. Since the sequence $\langle N_{0,\alpha} : \alpha < \alpha^* \rangle^\frown \langle N_{0,\alpha^*} \rangle$ is independent over M, $tp(a_\alpha, M_{0,\alpha}, M_{0,\alpha+1})$ does not fork over M. So by the transitivity (Definition 2.1), $tp(a_\alpha, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over M.

$\dashv$

This ends the proof of Proposition 4.17. $\dashv$

Proposition 4.18. *If $\langle M_\alpha : \alpha \leq \alpha^* + 1 \rangle$ is an increasing continuous sequence of models, then there is an increasing continuous sequence $\langle N_\alpha : \alpha \leq \alpha^* \rangle$ such that for $\alpha < \alpha^*$ $NF(M_\alpha, M_{\alpha+1}, N_\alpha, N_{\alpha+1})$ and $M_{\alpha^*+1} \preceq N_{\alpha^*}$.*

Proof. By the proof of Proposition 3.6. $\dashv$

Proposition 4.19. *The finitely independence has continuity. Equivalently: If*

(1) δ is a limit ordinal.
(2) $\langle M_\alpha : \alpha \leq \delta + 1 \rangle$ is increasing and continuous.
(3) $J \subset M_{\delta+1}$ and it is finite.
(4) For $\alpha < \delta$ J is independent in $(M, M_\alpha, M_{\delta+1})$.

then J is independent in $(M, M_\delta, M_{\delta+1})$.

Proof. By Proposition 4.18 there is another increasing continuous sequence $\langle N_\alpha : \alpha \leq \delta \rangle$ such that for $\alpha < \delta$ $NF(M_\alpha, M_{\alpha+1}, N_\alpha, N_{\alpha+1})$ and $M_{\delta+1} \preceq N_\delta$.

Take $\alpha < \delta$ with $J \subset N_\alpha$. Since J is independent in $(M, M_\alpha, M_{\delta+1})$ and $J \subseteq N_\alpha \preceq N_\delta \succeq M_{\delta+1}$, it follows that J is independent in (M, M_α, N_α). By the long transitivity in Fact 4.15, we have $NF(M_\alpha, N_\alpha, M_\delta, N_\delta)$. Therefore by Proposition 4.17 (where the models $M, M_\alpha, N_\alpha, M_\delta, N_\delta$ which appear here stands for the models M, M_0, M_1, M_2, M_3) J is independent in (M, M_δ, N_δ), hence in $(M, M_\delta, M_{\delta+1})$. $\dashv$

Finally, we prove $(*)$ of Proposition 4.14. Denote (*) for α^* by $(*)_{\alpha^*}$. We prove $(*)_{\alpha^*}$ by induction on α^*:

Case a: $\alpha^* = 0$. The conclusion is actually assumption (1).

Case b: $\alpha^* = \gamma + 1$. By the induction hypothesis $(*)_\gamma$ holds. We want to conclude that $J - \{a_\alpha : \alpha < \gamma\}$ is finitely independent in (M, M_γ, M^+). So we check the conditions:

(1): It is assumption (1).

(2): By assumption (2) $\{a_\alpha : \alpha < \alpha^*\} \subseteq J \subseteq M^+$. But $\{a_\alpha : \alpha < \gamma\} \subseteq \{a_\alpha : \alpha < \alpha^*\}$.

(3): By assumption (3) the sequence $\langle M_\alpha, a_\alpha : \alpha < \alpha^* \rangle^\frown \langle M_{\alpha^*} \rangle$ is independent over M by $K^{3,uq}$, namely the sequence $\langle M_\alpha, a_\alpha : \alpha < \gamma \rangle^\frown \langle M_\gamma, a_\gamma, M_{\alpha^*} \rangle$ is independent over M by $K^{3,uq}$. So the sequence $\langle M_\alpha, a_\alpha : \alpha < \gamma \rangle^\frown \langle M_\gamma \rangle$ is independent over M by $K^{3,uq}$.

(4): By assumption (4) $M_{\alpha^*} \preceq M^+$. But by assumption (3) $M_\gamma \preceq M_{\alpha^*}$. So

$M_\gamma \preceq M^+$.
Now by $(*)_\gamma$ the set $J - \{a_\alpha : \alpha < \gamma\}$ is finitely independent in (M, M_γ, M^+).

But by assumption 3, $(M_\gamma, M_{\alpha^*}, a_{\alpha^*}) \in K^{3,uq}$. So Proposition 4.12 (substituting $J - \{a_\alpha : \alpha < \gamma\}, M, M_\gamma, M^+, a_{\alpha^*}, M_{\alpha^*}$ instead of J, M, M_0, M^+, a, M_1 respectively), yields that $J - \{a_\alpha : \alpha < \alpha^*\}$ is finitely independent in (M, M_{α^*}, M^+).

Case c: α^* is limit. Let J^* be a finite subset of $J - \{a_\alpha : \alpha < \alpha^*\}$. We have to prove that J^* is independent in (M, M_{α^*}, M^+). For $\gamma < \alpha^*$ $J^* \subseteq J - \{a_\alpha : \alpha < \gamma\}$, so by $(*)_\gamma$, J^* is independent in (M, M_γ, M^+). Therefore by Proposition 4.19, J^* is independent in (M, M_{α^*}, M^+).

This ends the proof of Theorem 4.13. $\dashv$

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